# EFFECT OF SKEWED HEAT GENERATION AND NONUNIFORM HEAT TRANSFER AT THE SURFACE OF A HEAT EMITTING ROD ON ITS TEMPERATURE FIELD

L. N. Polyanin

UDC 536.242:621.039.517

Analytical solutions are obtained to problems concerning the temperature field of heat emitting rods with skewed sectional heat generation profiles and with an angular dependence of the heat transfer coefficient.

When considering the heat transfer at rod bundles in a turbulent stream of fluid (in the ducts of nuclear reactors, for instance), one often encounters the problem of determining the effects of a possibly skewed heat generation profile along an individual heat emitting element. Furthermore, the rod surface may also be subjected to nonuniform conditions of heat transfer to the ambient fluid (heat carrier), especially when surface boiling occurs around a part only of its perimeter.

In order to estimate the effect of a skewed heat generation profile and of zonal surface boiling, the author considers here each factor separately.

1. Effect of a Skewed Heat Generation Profile. As an example, we will consider a "plane" skew across the rod section.

With the dimensionless quantities

$$\rho = \frac{r}{r_0}, \; t(\rho, \varphi) = \frac{4\lambda T(\rho, \varphi)}{\bar{q}_v r_0^2}, \quad \mathrm{Bi} = \frac{kr_0}{\lambda},$$

where k denotes the coefficient of heat transfer from the rod surface to the heat carrier and the temperature of the latter is taken as the reference, we now have the equation of heat conduction written in polar coordinates

$$\frac{\partial^2 t}{\partial \rho^2} + \frac{1}{\rho} \cdot \frac{\partial t}{\partial \rho} + \frac{1}{\rho^2} \cdot \frac{\partial^2 t}{\partial \varphi^2} = -4(1 - \varepsilon\rho\cos\varphi) \tag{1.1}$$

with the conditions

$$\frac{\partial t}{\partial \rho} + \mathrm{Bi}\, t \Big|_{\rho=1} = 0, \tag{1.2}$$

$$\frac{\partial t}{\partial \varphi}\Big|_{\varphi=0} = \frac{\partial t}{\partial \varphi}\Big|_{\varphi=\pi} = 0, \tag{1.3}$$

$$t(0, \varphi) \neq \infty. \tag{1.4}$$

Multiplying the original equation (1.1) by $\cos p\varphi$ and integrating the result from 0 to $\pi$ (a finite cosine transformation), with (1.3) taken into account, we obtain for the spectrum of temperature

$$\theta_p(\rho) = \int_0^\pi t(\rho, \varphi) \cos p\varphi \, d\varphi, \; p = 0, 1, 2, \ldots$$

the following equation:

$$\frac{d^2\theta_p}{d\rho^2} + \frac{1}{\rho} \cdot \frac{d\theta_p}{d\rho} - \frac{p^2}{\rho^2}\theta_p = \begin{cases} -4\pi, & p = 0, \\ 2\pi\varepsilon\rho, & p = 1, \\ 0, & p = 2, 3, 4, \ldots \end{cases} \tag{1.5}$$

Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 24, No. 6, pp. 1118-1123, June, 1973. Original article submitted June 12, 1972.

Considering that the solution is bounded at the center of (1.4), we then find that

$$\left.\begin{aligned}\theta_0(\rho) &= C_0 - \pi\rho^2,\\ \theta_1(\rho) &= C_1\rho + \frac{\pi\varepsilon}{4}\rho^3,\\ \theta_p(\rho) &= C_p\rho^p,\ p = 2,\ 3,\ 4,\ \ldots\end{aligned}\right\} \tag{1.6}$$

The solution $t(\rho, \varphi)$ is determined in terms of the $\theta_p(\rho)$ spectrum through a series [1]:

$$t(\rho, \varphi) = \frac{1}{\pi}\theta_0(\rho) + \frac{2}{\pi}\sum_{p=1}^{\infty}\theta_p(\rho)\cos p\varphi. \tag{1.7}$$

Inserting expression (1.7) into the boundary condition (1.2), we find the values of the constants $C_p$:

$$\left.\begin{aligned}C_0 &= \frac{2\pi}{\mathrm{Bi}}\left(1 + \frac{\mathrm{Bi}}{2}\right),\\ C_1 &= -4\pi\varepsilon\,\frac{3+\mathrm{Bi}}{1+\mathrm{Bi}},\\ C_p &= 0,\ p = 2,\ 3,\ 4,\ \ldots\end{aligned}\right\} \tag{1.8}$$

In this way, the sought solution will appear as

$$t(\rho, \varphi) = \left(1 + \frac{2}{\mathrm{Bi}}\right) - \rho^2 - \frac{\varepsilon}{2}\left(\frac{3+\mathrm{Bi}}{1+\mathrm{Bi}} - \rho^2\right)\rho\cos\varphi. \tag{1.9}$$

The difference between maximum and minimum thermal flux from the surface of a heat emitting element (with the dimensionless mean thermal flux $\bar{q}_s = 2$) will be

$$\Delta q_s^{\max} = q_s^{\max} - q_s^{\min} = 2\varepsilon\,\frac{\mathrm{Bi}}{1+\mathrm{Bi}}, \tag{1.10}$$

while the difference between maximum and minimum surface temperature of a heat emitting element will be

$$\Delta t_s^{\max} = t_s^{\max} - t_s^{\min} = 2\varepsilon\,\frac{1}{1+\mathrm{Bi}}. \tag{1.11}$$

The point $(\rho^*, \varphi^*)$ of a heat emitting element which is at the maximum temperature can be found by a simultaneous solution of equations $\partial t/\partial\rho = 0$ and $\partial t/\partial\varphi = 0$, which will yield its coordinates:

$$\rho^* = \frac{2}{3\varepsilon}\left(\sqrt{1 + \frac{3\varepsilon^2}{4}\,\frac{3+\mathrm{Bi}}{1+\mathrm{Bi}}} - 1\right) \approx \frac{\varepsilon}{4}\,\frac{3+\mathrm{Bi}}{1+\mathrm{Bi}},$$

$$\varphi^* = \pi.$$

The expression for the maximum temperature $t^*$ will be

$$t^* \approx \left(1 + \frac{2}{\mathrm{Bi}}\right) - \frac{\varepsilon^4}{128}\left(\frac{3+\mathrm{Bi}}{1+\mathrm{Bi}}\right)^3, \tag{1.12}$$

i.e., a "plane" skew of the heat source (with the heat generation per unit rod length maintained) lowers the maximum temperature of a heat emitting element by the amount $\varepsilon^4/128((3+\mathrm{Bi})/(1+\mathrm{Bi}))^3$.

For a numerical evaluation of these results, let us consider a heat emitting element 6 mm in diameter ($r_0 = 3$ mm) with a $\delta = 0.3$ mm thick shell. Assuming the thermal conductivity of the rod material to be $\lambda = 35$ W/m·°C and that of the shell material to be $\lambda_{sh} = 20$ W/m·°C, with a heat transfer coefficient at the wall surface $\alpha = 35{,}000$ W/m$^2$·°C and $\varepsilon = (q_v^{\max} - \bar{q}_v)/\bar{q}_v = 0.2$, we obtain Bi ≈ 2,00, $\Delta q_s^{\max}/\bar{q}_s \approx 0.133$, $\Delta t_s^{\max} \approx 0.133$, and $\rho^* \approx 0.083$.

The change in the maximum rod temperature is negligible and only about $0.35 \cdot 10^{-4}$.

2. Effect of a Nonuniform Heat Transfer Coefficient around the Perimeter of a Heat Emitting Element. We will now consider the Biot number, which characterizes the heat transfer from a rod to the ambient medium, and its angular variation in the form

$$\mathrm{Bi}(\varphi) = \overline{\mathrm{Bi}} - \frac{\Delta\mathrm{Bi}}{2}\cos\varphi, \tag{2.1}$$

where

$$\bar{\text{Bi}} = \frac{1}{2}(\text{Bi}^{\max} + \text{Bi}^{\min}), \ \ \Delta\,\text{Bi} = \text{Bi}^{\max} - \text{Bi}^{\min}.$$

In the case of a heat emitting element with a uniform cross section, we have for the temperature spectrum $\theta_p(\rho)$ the following equation:

$$\frac{d^2\theta_p}{d\rho^2} + \frac{1}{\rho}\cdot\frac{d\theta_p}{d\rho} - \frac{p^2}{\rho^2}\theta_p = \begin{cases} -4\pi, & p = 0, \\ 0, & p = 1, 2, 3, \ldots \end{cases} \tag{2.2}$$

Conditions (1.3) and (1.4) yield

$$\left.\begin{aligned} \theta_0(\rho) &= C_0 - \pi\rho^2, \\ \theta_p(\rho) &= C_p\rho^p, \ p = 1, 2, 3, \ldots \end{aligned}\right\} \tag{2.3}$$

Inserting the solution

$$t(\rho, \varphi) = \frac{1}{\pi}C_0 - \rho^2 + \frac{2}{\pi}\sum_{p=1}^{\infty} C_p\rho^p\cos p\varphi \tag{2.4}$$

into the boundary condition (1.2), with (2.1) also taken into account, we obtain the following system of equations for the coefficients $C_p$ ($\beta = \Delta\text{Bi}/2$):

$$\left.\begin{aligned} &\frac{\bar{\text{Bi}}}{\beta}C_0 - C_1 = \frac{\pi}{\beta}(2 + \bar{\text{Bi}}), \\ &C_0 - \frac{2}{\beta}(1 + \bar{\text{Bi}})\,C_1 + C_2 = \pi, \\ &C_1 - \frac{2}{\beta}(1 + \bar{\text{Bi}})\,C_2 + C_3 = 0, \\ &C_{p-2} - \frac{2(p-1)}{\beta}C_{p-1} + C_p = 0, \ \ p = 4, 5, 6, \ldots \end{aligned}\right| \tag{2.5}$$

These equations for $C_p$, beginning with the fourth one, constitute a recurrent series and their solution is a linear combination of cylindrical Bessel and Neumann functions of the p-th order:

$$C_p = aJ_p(\beta) + bN_p(\beta).$$

Retaining only a finite number of terms in the sum of series (2.4), and letting $C_p = 0$, we obtain an equation which relates coefficients $a$ and b:

$$\frac{b}{a} = -\frac{J_P(\beta)}{N_P(\beta)}.$$

With a sufficiently large p = P one may use the "tangents approximation" for functions $J_p(\beta)$ and $N_p(\beta)$. Letting $p/\beta = \text{ch}\,\mu$, we have [2]

$$J_p\left(\frac{p}{\text{ch}\,\mu}\right) \sim \frac{\exp(p\,\text{th}\,\mu - p\mu)}{\sqrt{2\pi p\,\text{th}\,\mu}}\left[1 + o\left(\frac{1}{p}\right)\right],$$

$$N_p\left(\frac{p}{\text{ch}\,\mu}\right) \sim \frac{\exp(p\mu - p\,\text{th}\,\mu)}{\sqrt{\frac{\pi}{2}p\,\text{th}\,\mu}}\left[1 - o\left(\frac{1}{p}\right)\right].$$

Thus, for large values of p we have the asymptotic proportionality

$$\frac{J_p(\beta)}{N_p(\beta)} \sim \frac{1}{2}\exp\left[2(p\,\text{th}\,\mu - p\mu)\right] \sim \frac{1}{2}\exp(-2p\mu)$$

$$= \frac{1}{2}\exp\left[-2p\,\text{Arch}\,\frac{p}{\beta}\right] \approx \frac{1}{2}\exp\left(-2p\ln\frac{2p}{\beta}\right) = \frac{1}{2}\,\frac{1}{\left(\frac{2p}{\beta}\right)^{2p}}.$$

Consequently, as p tends toward infinity, b/$a$ approaches zero fast. Therefore,

$$C_p(\beta) = aJ_p(\beta), \ \ p = 2, 3, 4, \ldots \tag{2.6}$$

Letting p = 2 and p = 3 in (2.6), we will obtain the values of coefficients $C_2$ and $C_3$, respectively, in both the second and the third equation of system (2.5) accurately down to the constant $a$, whereupon the first three equations of system (2.5) will yield the values of the unknowns $a$, $C_0$, and $C_1$:

$$a = -\frac{2\pi\beta - [2\,\overline{\mathrm{Bi}}\,(1+\overline{\mathrm{Bi}}) - \beta^2]}{\beta\,\overline{\mathrm{Bi}}\,J_2(\beta)}, \tag{2.7}$$

$$C_0 = \pi\left(1 + \frac{2}{\overline{\mathrm{Bi}}}\right) + \frac{\beta}{\overline{\mathrm{Bi}}}C_1, \tag{2.8}$$

$$C_1 = \frac{2\pi\beta R(\beta)}{\beta\,\overline{\mathrm{Bi}} + [2\overline{\mathrm{Bi}}\,(1+\overline{\mathrm{Bi}}) - \beta^2]\,R(\beta)}, \tag{2.9}$$

where

$$R(\beta) = \frac{2(1+\overline{\mathrm{Bi}})}{\beta} - \frac{J_3(\beta)}{J_2(\beta)}. \tag{2.10}$$

As is evident according to (2.4), the expression for the thermal flux from the surface of a heat emitting element becomes

$$q_s(\varphi) = 2 - \frac{2}{\pi}\sum_{p=1}^{\infty} pC_p \cos p\varphi.$$

Consequently,

$$\Delta q_s^{\max} = \frac{4}{\pi}\sum_{p=0}^{\infty}(2p+1)\,C_{2p+1}. \tag{2.11}$$

The maximum skew of temperature around the rod perimeter is

$$\Delta t_s^{\max} = \frac{4}{\pi}\sum_{p=0}^{\infty} C_{2p+1}. \tag{2.12}$$

Using the equalities

$$\sum_{p=0}^{\infty}(2+1)\,J_{2p+1}(\beta) = \frac{1}{2}\beta, \tag{2.13}$$

$$\sum_{p=0}^{\infty} J_{2p+1}(\beta) = \frac{1}{2}\int_0^{\beta} J_0(x)\,dx = \beta J_0(\beta) - \frac{1}{2}\pi\beta\,[J_0(\beta)H_1(\beta) - J_1(\beta)H_0(\beta)], \tag{2.14}$$

where $H_0(\beta)$ and $H_1(\beta)$ are Struve functions [2], we may replace (2.11) and (2.12) by

$$\Delta q_s^{\max} = \frac{4}{\pi}\left\{C_1 + a\left[\frac{1}{2}\beta - J_1(\beta)\right]\right\}, \tag{2.15}$$

$$\Delta t_s^{\max} = \frac{4}{\pi}\left\{C_1 + a\left[\frac{1}{2}\int_0^{\beta} J_0(x)\,dx - J_1(\beta)\right]\right\}. \tag{2.16}$$

For the preceding sample with $\mathrm{Bi}^{\max} = 4$ (which corresponds to a heat transfer coefficient $\alpha^{\max} \approx 150{,}000\ \mathrm{W/m^2\cdot{}^\circ C}$) and $\mathrm{Bi}^{\min} = 2$, we have

$$\Delta q_s^{\max} \approx 0.32;\quad \Delta t_s^{\max} \approx 0.33.$$

In order to convert to dimensional values of thermal flux and temperature, it is necessary to bring in the equalities

$$Q_s = \frac{\bar{q}_v r_0}{4} q_s,\quad T = \frac{\bar{q}_v r_0^2}{4\lambda} t.$$

We note that, letting $C_3 = C_4 = \cdots = 0$ (approximate solution) in Eq. (2.5), we have

$$\Delta q_s^{\max} = \Delta t_s^{\max} \approx \frac{16\left(1 + \frac{\overline{\mathrm{Bi}}}{2}\right)}{\overline{\mathrm{Bi}}\left\{2\left(1 + \frac{\overline{\mathrm{Bi}}}{2}\right)\left[\frac{2}{\beta}(1+\overline{\mathrm{Bi}}) - \frac{\beta}{\overline{\mathrm{Bi}}}\right] - \frac{\beta}{2}\right\}}, \tag{2.17}$$

which yields

$$\Delta q_s^{max} = \Delta t_s^{max} \approx 0,35.$$

## NOTATION

| | |
|---|---|
| $r_0$ | is the radius of a cylindrical heat emitting rod; |
| $\lambda$ | is the thermal conductivity of the rod material; |
| k | is the coefficient of heat transfer from the rod surface to the ambient medium; |
| Bi | is the Biot number; |
| $\bar{q}_v$ | is the mean emitted thermal power density in a rod; |
| t | is the dimensionless temperature; |
| T | is the actual temperature; |
| $q_s$ | is the dimensionless thermal flux density at the rod surface; |
| $Q_s$ | is the actual thermal flux density at the rod surface; |
| $\rho$ and $\varphi$ | are the polar coordinates; |
| $\theta$ | is the cosine transform of the temperature; |
| $\varepsilon$ | is the relative skew of the sectional heat generation profile; |
| $J_p$ | is the cylindrical Bessel function of the p-th order; |
| $N_p$ | is the cylindrical Neumann function of the p-th order; |
| $H_0$ and $H_1$ | are Struve functions. |

## LITERATURE CITED

1. K. J. Tranter, Integral Transformations in Mathematical Physics [Russian translation], GITTL, Moscow (1956).
2. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], GIMFL, Moscow (1963).